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The interaction between a single two-level atom coupled to an N-level quantum system through three couplings

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ABSTRACT

We introduce the problem of three types of interaction between an N-level quantum system and a two-level atom where three coupling parameters are involved. The system can be deduced from the Heisenberg chain. The canonical transformation is used to remove two coupling parameters from the system and consequently it is reduced to atom–atom interaction. The wave function is calculated using the evolution operator and hence we have managed to obtain the expectation value of some dynamical operators. During our study of the atomic inversion we noted that the collapses period is shifted up when we take the effect of λ_2 into consideration. While it is shifted up and down in the presence of λ_3. The atomic angle plays a crucial role for controlling the degree of entanglement. For the variance squeezing we noted that the coupling parameter λ_2 shows amounts of squeezing more than the case of λ_3. Similar behavior is noted for the entropy squeezing.

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1. Introduction

The problem of the interaction is regarded as one of the fundamental problems in the field of quantum optics. It is well known that there are three different kinds of the interaction; one of them is the field–field interaction [1–5] and another is the atom–field interaction [6–12], while the third type is the atom–atom interaction [13–18]. In fact each one of these types, describes certain kinds of the phenomena, however it is possible to transfer one to another, for more details one can see, Refs. [19,20]. In this context we may mention the following physical systems; collective spontaneous emission from an ensemble of N two-level in the single-photon Dicke super radiance [21]. A quantum model based on quasiparticles is used to describe electron–ion interaction in a non-ideal plasma. The quasiparticles interact with a quantized field through the frequency converter type with a Kerr-like term to describe the interaction between the quasiparticles. This model has been used to show the variation line profile with the density in a weakly non-ideal plasma [22]. The interaction of two solutions with absorbing three-level atom has been treated to investigate the existence solitary wave pair [23]. The autocorrelation function is derived for the field emitted by a Rydberg atom in a cavity weakly excited in the strong coupling regime in the resonance case [24]. Furthermore the atoms–field interaction as well as the atoms–atoms interaction attracted much attention in this field. This is due to the appearance of the fields of quantum teleportation, quantum cryptography, quantum computing and quantum information which have been built up on the concept of the entanglement [25–27]. Doubtless quantum entanglement represents one of the corner stones of the theory of quantum mechanics and is of fundamental interest in modern physics. Several Hamiltonian models have been introduced in the literature to discuss the phenomena which appeared as a result of the interaction between the atoms and the fields as well as between the atoms and atoms. One of these models is Jaynes–Cummings model (JCM) which represents the interaction between a two-level atom and an electromagnetic field within a cavity. Also we can see Tavis–Cummings model (TCM) which generalizes JCM to deal with many two-level atoms. The Hamiltonian which describes such a model is given by

$$\frac{\hat{H}}{\hbar} = \omega \hat{a}^{\dagger}\hat{a} + \frac{\omega_0}{2}\hat{J}_z + \lambda\left(\hat{a}\hat{J}_{+} + \hat{a}^{\dagger}\hat{J}_{-}\right), \tag{1}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators of the field mode that satisfy the relation $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$. ω and ω_0 are the field and the atomic frequencies, respectively, while $\hat{J}_{\pm}$ and $\hat{J}_z$ are the collective angular momentum operators which satisfy the commutation relations

$$\left[\hat{J}_{\pm}, \hat{J}_z\right] = \mp\hat{J}_{\pm} \qquad \left[\hat{J}_{-}, \hat{J}_{+}\right] = -2\hat{J}_z. \tag{2}$$

On the other hand the coherent control of a two-electron spin state in a coupled quantum dot was achieved experimentally, in which the coupling mechanism is the Heisenberg exchange interaction between the electron spins [28–31]. In this context one can find the proposals for an experimental scheme to create, detect and control entangled spin states in coupled quantum dots as introduced in Ref. [32]. In the solid state systems there is a special interest where an integrated network can perform quantum computing algorithms at a large scale. For instance the semi conductor quantum dot is considered as one of the most promising candidates to play the role of a qubit, see for example Refs. [33–37].

The main idea is to use the spin S of the valence electron on a single quantum dot as a two-state quantum system which gives rise to a well-defined qubit. As a result of this interest many efforts have been devoted to investigate the interacting Heisenberg spin chain as it represents a very reliable model for constructing quantum computing schemes in different solid state systems, as well as for being a very rich model for studying the novel physics of localized spin systems [36–40]. The most familiar model which can be used to describe such cases, is given by

$$\hat{H}(t) = \sum_{j=1}^{2} \mathcal{B}_j(t)\hat{S}_z^{(j)} + \sum_{i=x,y,z} \mathcal{A}_i \hat{S}_i^{(1)}\hat{S}_i^{(2)} \tag{3}$$

where $\hat{S}_z^{(j)} = \frac{1}{2}\hat{\sigma}_i$, ($i = x, y, z$ and $j = 1, 2$) are the spin $\frac{1}{2}$-operators and $\hat{\sigma}_i$ are the usual Pauli operators. $\mathcal{A}_i$ are the strengths of the Heisenberg interactions in the x, y and z directions, respectively, while $\mathcal{B}_j(t)$ are the external time-dependent magnetic fields. In fact this Hamiltonian describes two coupled qubits through anisotropic Heisenberg *XYZ* interaction in a nonuniform magnetic field in the z-direction. Recently many efforts can be seen to find the exact time-dependent dynamical operators, which would enable us to discuss many phenomena may occur as a result of such kind of the interaction. However, it is due to the nonlinearity as well as the explicit time-dependence of the system that it is very hard to get such kind of solution.

In the present communication we modify the above Hamiltonian to include the N-level atom. Furthermore we restrict our treatment for the case in which the coupling parameters are constants. In this case we can write the Hamiltonian in the form

$$\frac{\hat{H}}{\hbar} = \eta\hat{\sigma}_z + \mu\hat{J}_z + \lambda_1\left(\hat{\sigma}_- - \hat{\sigma}_+\right)\left(\hat{J}_+ - \hat{J}_-\right) + \lambda_2\left(\hat{J}_+ + \hat{J}_-\right) + \lambda_3\left(\hat{\sigma}_- + \hat{\sigma}_+\right) \tag{4}$$

where $\lambda_i, i = 1, 2, 3$ are three different coupling parameters, while $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are the two-level atom Pauli operators that satisfy the relation

$$\left[\hat{\sigma}_+, \hat{\sigma}_-\right] = \hat{\sigma}_z, \qquad \left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] = \pm 2\hat{\sigma}_\pm, \tag{5}$$

while the operators $J_\pm$ and J_z are the N-level atom operators which obey the relations

$$\left[\hat{J}_\pm, \hat{J}_z\right] = \mp\hat{J}_\pm, \qquad \left[\hat{J}_-, \hat{J}_+\right] = -2\hat{J}_z. \tag{6}$$

As one sees the above Hamiltonian represents the interaction between pair of atoms; the first one is a two-level and the second is an N-level atom under the effect of two different external electromagnetic fields.

By making comparison between Eqs. (3) and (4), it is noted that the first two terms in (4) are the same as the first in (3), while the second term of Eq. (3) can be produced from the second term of Eq. (4). The last two terms in (4) are regarded as external electromagnetic fields (as previously mentioned). In what follows we shall find the time-dependent dynamical operators from which we are able to discuss some statistical properties. For this reason we devote the next two sections to do so. In Section 4 we consider the atomic population, while in Section 5 we discuss the degree of entanglement through the linear entropy which is followed by the variance and the entropy squeezing in Sections 6 and 7, respectively. Our conclusion is given in Section 8.

2. The wave function

In order to discuss the dynamics of the system we have to find the solution of the Heisenberg equations of motion or to obtain the wave function using the Schrödinger picture. Alternatively one may use the evolution operator to reach the same goal. In the present paper we employ the evolution operator to find the wave function. However, it is due to the complications of the system, it is unlikely to reach this aim. Therefore, we introduce the following transformation:

$$\begin{pmatrix} \hat{J}_- \\ \hat{J}_+ \\ \hat{J}_z \end{pmatrix} = \begin{pmatrix} \cos^2\alpha & -\sin^2\alpha & \frac{1}{2}\sin 2\alpha \\ -\sin^2\alpha & \cos^2\alpha & \frac{1}{2}\sin 2\alpha \\ -\sin 2\alpha & -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \hat{R}_- \\ \hat{R}_+ \\ \hat{R}_z \end{pmatrix}, \tag{7}$$

where $\alpha = \frac{1}{2}\tan^{-1}\left(\lambda_2/\mu\right)$ and $\hat{R}_{\pm,z}$ satisfy the relations

$$\left[\hat{R}_z, \hat{R}_\pm\right] = \pm\hat{R}_\pm, \qquad \left[\hat{R}_-, \hat{R}_+\right] = -2\hat{R}_z. \tag{8}$$

Furthermore, we introduce the transformation

$$\begin{pmatrix}\hat{\sigma}_- \\ \hat{\sigma}_+ \\ \hat{\sigma}_z\end{pmatrix} = \begin{pmatrix}\cos^2\beta & -\sin^2\beta & \frac{1}{2}\sin 2\beta \\ -\sin^2\beta & \cos^2\beta & \frac{1}{2}\sin 2\beta \\ -\sin 2\beta & -\sin 2\beta & \cos 2\beta\end{pmatrix}\begin{pmatrix}\hat{g}_- \\ \hat{g}_+ \\ \hat{g}_z\end{pmatrix}, \tag{9}$$

where $\beta = \frac{1}{2}\tan^{-1}(2\lambda_3/\eta)$ and $\hat{g}_{\pm,z}$ have the same properties of $\hat{\sigma}_{\pm,z}$. Now if we substitute the transformations given by Eqs. (7) and (9) into the Hamiltonian (4), then we obtain

$$\frac{\hat{H}}{\hbar} = \bar{\eta}\hat{g}_z + \bar{\mu}\hat{R}_z + \lambda_1\left(\hat{g}_-\hat{R}_+ + \hat{g}_+\hat{R}_-\right), \tag{10}$$

where we have applied the rotating wave approximation (RWA) on the transformation operators and defined the augmented frequencies

$$\bar{\eta} = \sqrt{\eta^2 + 4\lambda_3^2}, \qquad \bar{\mu} = \sqrt{\mu^2 + 4\lambda_2^2}. \tag{11}$$

From the Heisenberg equations of motion we have

$$\frac{d}{dt}\hat{g}_z = -i\lambda_1\left(\hat{R}_-\hat{g}_+ - \hat{g}_-\hat{R}_+\right), \qquad \frac{d}{dt}\hat{R}_z = i\lambda_1\left(\hat{R}_-\hat{g}_+ - \hat{g}_-\hat{R}_+\right), \tag{12}$$

from which we have

$$\hat{M} = \hat{g}_z + \hat{R}_z, \tag{13}$$

which is a constant of motion. In this way we can write the Hamiltonian (10) in the form

$$\frac{\hat{H}}{\hbar} = \bar{\mu}\hat{M} + \hat{C} \tag{14}$$

and

$$\hat{C} = \Delta\hat{g}_z + \lambda_1\left(\hat{g}_-\hat{R}_+ + \hat{g}_+\hat{R}_-\right), \quad \Delta = (\bar{\eta} - \bar{\mu}) \tag{15}$$

where $\hat{C}$ is also a constant of motion. Since $\hat{M}$ and $\hat{C}$ are constants of motion then the commutation relation $[\hat{M}, \hat{C}] = 0$ is always satisfied and consequently both of them commute with the Hamiltonian (14). Using this fact we can write the time-evolution operator as follows

$$\begin{aligned}\hat{U}(t) &= \exp\left(-\frac{\mathrm{i}\hat{H}}{\hbar}t\right) \\ &= \exp(-\mathrm{i}\bar{\mu}\hat{M}t)\exp(-\mathrm{i}\hat{C}t) = \begin{bmatrix}\hat{F}_{11}(t) & \hat{F}_{12}(t) \\ \hat{F}_{21}(t) & \hat{F}_{22}(t)\end{bmatrix},\end{aligned} \tag{16}$$

where we have used the abbreviations

$$\begin{aligned}\hat{F}_{11}(t) &= \exp\left[-\mathrm{i}\bar{\mu}\left(\hat{R}_z + \frac{1}{2}\right)t\right]\left(\cos\hat{\eta}_1 t - \frac{\mathrm{i}\Delta}{2}\frac{\sin\hat{\eta}_1 t}{\hat{\eta}_1}\right), \\ \hat{F}_{12}(t) &= -\mathrm{i}\mu_2\exp\left[-\mathrm{i}\bar{\mu}\left(\hat{R}_z + \frac{1}{2}\right)t\right]\frac{\sin\hat{\eta}_1 t}{\hat{\eta}_1}\hat{R}_-, \\ \hat{F}_{21}(t) &= -\mathrm{i}\mu_2\exp\left[-\mathrm{i}\bar{\mu}\left(\hat{R}_z - \frac{1}{2}\right)t\right]\frac{\sin\hat{\eta}_2 t}{\hat{\eta}_2}\hat{R}_+, \\ \hat{F}_{22}(t) &= \exp\left[-\mathrm{i}\bar{\mu}\left(\hat{R}_z - \frac{1}{2}\right)t\right]\left(\cos\hat{\eta}_2 t + \frac{\mathrm{i}\Delta}{2}\frac{\sin\hat{\eta}_2 t}{\hat{\eta}_2}\right)\end{aligned} \tag{17}$$

and defined

$$\hat{\eta}_1^2 = \Delta^2 + \lambda_1^2 \hat{R}_- \hat{R}_+, \qquad \hat{\eta}_1^2 = \Delta^2 + \lambda_1^2 \hat{R}_+ \hat{R}_-. \tag{18}$$

This in fact would help us to calculate the expectation values of the dynamical operators. Now suppose we consider the single two-level atom is initially in the superposition state

$$|\vartheta, \varphi\rangle = \cos(\vartheta)\,|e\rangle + e^{i\varphi}\sin(\vartheta)\,|g\rangle, \tag{19}$$

where ϑ and φ are the coherence and the relative phase angles respectively. The transformations (9) introduce the new states $|+\rangle$ and $|-\rangle$ eigenstates of $\hat{S}_z$ such that

$$|+\rangle = \cos\beta|e\rangle + \sin\beta|g\rangle, \qquad |-\rangle = \cos\beta|g\rangle - \sin\beta|e\rangle. \tag{20}$$

Therefore we can write $|\vartheta, \varphi\rangle$ in the form

$$|\vartheta, \varphi\rangle = \left[\cos\beta\cos\vartheta + e^{i\varphi}\sin\beta\sin\vartheta\right]|+\rangle - \left[\sin\beta\cos\vartheta - e^{i\varphi}\cos\beta\sin\vartheta\right]|-\rangle. \tag{21}$$

Furthermore, we consider a coherent spin state as initial state for the other atoms [41],

$$|\Theta, \Phi\rangle = \sum_{m=-r}^{r} K_m^r(\Theta, \Phi)|m, , r\rangle, \tag{22}$$

where

$$K_m^r(\Theta, \Phi) = \sqrt{C_{r+m}^{2r}}\exp[i(r-m)\Phi]\cos^{r+m}\left(\frac{\Theta}{2}\right)\sin^{r-m}\left(\frac{\Theta}{2}\right), \tag{23}$$

while C_m^n are the binomial coefficients. Note that the states $|m, r\rangle$ are the eigenfunction of R_z and satisfy

$$\begin{aligned}
\hat{R}_+|m, r\rangle &= \sqrt{(r-m)(r+m+1)}|m+1, r\rangle, \\
\hat{R}_-|m, r\rangle &= \sqrt{(r-m+1)(r+m)}|m-1, r\rangle, \\
\hat{R}_z|m, r\rangle &= m|m, r\rangle.
\end{aligned} \tag{24}$$

In what follows we use the evolution operator $\hat{U}(t)$ to calculate the wave function as well as the expectation values of some variable. This is seen in the next section.

3. The expectation values

To reach our goal we devote the present section to calculate the expectation values for some dynamical operators which can be used to discuss some statistical properties of the system. To do so we use the above states and we define the initial state of the system in the form $|\Psi(0)\rangle = |\vartheta, \varphi\rangle \otimes |\Theta, \Phi\rangle$.

Now we are able to evaluate different moments of the involving operators in the present system. To do so we first write the state $|\Psi(t)\rangle$ at $t > 0$ in the form

$$|\psi(t)\rangle = \sum_m R_m^r\left[\hat{D}(t)|m, r, +\rangle - \hat{T}(t)|m, r, -\rangle\right] \tag{25}$$

where

$$\hat{D}(t) = c_1\hat{F}_{11} - s_1\hat{F}_{12}, \qquad \hat{T}(t) = s_1\hat{F}_{22} - c_1\hat{F}_{21}, \tag{26}$$

and

$$c_1 = \cos\beta\cos\theta + e^{i\bar{\phi}}\sin\beta\sin\theta, \qquad s_1 = \sin\beta\cos\theta - e^{i\bar{\phi}}\cos\beta\sin\theta. \tag{27}$$

Having obtained the wave function, therefore we are in a position to calculate the expectation values of the operators $\hat{\sigma}_x$; $\hat{\sigma}_y$ and $\hat{\sigma}_z$: after some calculations we have the expressions

$$\begin{aligned}\left\langle\hat{\sigma}_z(t)\right\rangle &= \left\langle\hat{g}_z(t)\right\rangle\cos 2\beta - \left\langle\hat{g}_x(t)\right\rangle\sin 2\beta \\ \left\langle\hat{\sigma}_x(t)\right\rangle &= \left\langle\hat{g}_x(t)\right\rangle\cos 2\beta + \left\langle\hat{g}_z(t)\right\rangle\sin 2\beta \\ \left\langle\hat{\sigma}_y(t)\right\rangle &= \left\langle\hat{g}_y(t)\right\rangle\end{aligned} \tag{28}$$

where

$$\begin{aligned}\left\langle\hat{g}_z(t)\right\rangle &= \langle\hat{D}^\dagger(t)\hat{D}(t)\rangle - \langle\hat{T}^\dagger(t)\hat{T}(t)\rangle \\ \left\langle\hat{g}_x(t)\right\rangle &= -\langle\hat{D}^\dagger(t)\hat{T}(t)\rangle - \langle\hat{T}^\dagger(t)\hat{D}(t)\rangle \\ \left\langle\hat{g}_y(t)\right\rangle &= i\left[\langle\hat{T}^\dagger(t)\hat{D}(t)\rangle - \langle\hat{D}^\dagger(t)\hat{T}(t)\rangle\right]\end{aligned} \tag{29}$$

and we have defined

$$\begin{aligned}\langle\hat{D}^\dagger(t)\hat{D}(t)\rangle &= \sum_{m=-j}^{j}|K_m^j(\Theta,\Phi)|^2\left[|c_1|^2|F_{11}(m,t)|^2 + |s_1|^2|E_2(m,t)|^2\right] \\ &\quad - \sum_{m=-j}^{j-1}\left(K_m^j(\Theta,\Phi)\right)^* K_{m+1}^j(\Theta,\Phi)c_1^*s_1F_{11}^*(m,t)E_1(m,t) \\ &\quad - \sum_{m=-j}^{j-1}\left(K_{m+1}^j(\Theta,\Phi)\right)^* K_m^j(\Theta,\Phi)s_1^*c_1F_{11}(m,t)E_1(m,t)\end{aligned} \tag{30}$$

and

$$\begin{aligned}\langle\hat{T}^\dagger(t)\hat{T}(t)\rangle &= \sum_{m=-j}^{j}|K_m^j(\Theta,\Phi)|^2[|s_1|^2|F_{22}(m,t)|^2 + |c_1|^2|E_1(m,t)|^2] \\ &\quad + \sum_{m=-j}^{j-1}\left(K_{m+1}^j(\Theta,\Phi)\right)^* K_m^j(\Theta,\Phi)s_1^*c_1F_{11}E_1(m,t) \\ &\quad + \sum_{m=-j}^{j-1}\left(K_m^j(\Theta,\Phi)\right)^* K_{m+1}^j(\Theta,\Phi)c_1^*s_1F_{11}^*(m,t)E_1(m,t),\end{aligned} \tag{31}$$

while

$$\begin{aligned}\langle\hat{D}^\dagger(t)\hat{T}(t)\rangle &= \exp(-i\bar{\mu}t)\left(\sum_{m=-j}^{j}|K_m^j(\Theta,\Phi)|^2c_1^*s_1F_{11}^*(m,t)F_{22}(m,t)\right. \\ &\quad + \sum_{m=-j+1}^{j}\left(K_m^j(\Theta,\Phi)\right)^* K_{m-1}^j(\Theta,\Phi)|c_1|^2F_{11}^*(m,t)E_2(m,t) \\ &\quad - \sum_{m=-j}^{j-1}\left(K_{m+1}^j(\Theta,\Phi)\right)^* K_m^j(\Theta,\Phi)|s_1|^2F_{22}(m,t)E_1(m,t) \\ &\quad \left. - \sum_{m=-j}^{j-2}\left(K_{m+2}^j(\Theta,\Phi)\right)^* K_m^j(\Theta,\Phi)s_1^*c_1E_1(m,t)E_1(m+1,t)\right)\end{aligned} \tag{32}$$

and its complex

$$
\begin{aligned}
\langle \hat{T}^{\dagger}(t)\hat{D}(t)\rangle = \exp(i\bar{\mu}t)\Bigg(&\sum_{m=-j}^{j} |K_m^j(\Theta,\Phi)|^2 s_1^* c_1 F_{22}^*(m,t)F_{11}(m,t)\\
&+\sum_{m=-j+1}^{j}\left(K_{m-1}^j(\Theta,\Phi)\right)^* K_m^j(\Theta,\Phi)|c_1|^2 F_{11}(m,t)E_2(m,t)\\
&-\sum_{m=-j}^{j-1}\left(K_m^j(\Theta,\Phi)\right)^* K_{m+1}^j(\Theta,\Phi)|s_1|^2 F_{22}^*(m,t)E_1(m,t)\\
&-\sum_{m=-j}^{j-2}\left(K_m^j(\Theta,\Phi)\right)^* K_{m+1}^j(\Theta,\Phi)c_1^* s_1 E_1(m,t)E_1(m+1,t)\Bigg).
\end{aligned}
\tag{33}
$$

In the above equations we used the abbreviations

$$
\begin{aligned}
&F_{11}(m,t) = \left(\cos\eta_1 t - \frac{i\Delta}{2}\frac{\sin\eta_1 t}{\eta_1}\right), \qquad F_{22}(m,t) = \left(\cos\eta_2 t + \frac{i\Delta}{2}\frac{\sin\eta_2 t}{\eta_2}\right)\\
&E_1(m,t) = \left(\frac{\nu_1(m)\sin\eta_1(m)t}{\mu_1(m)}\right), \qquad E_2(m,t) = \left(\frac{\nu_2(m)\sin\eta_2(m)t}{\eta_2(m)}\right)\\
&\nu_1(m) = \sqrt{(r-m)\,(r+m+1)}, \qquad \nu_2(m) = \sqrt{(r+m)\,(r-m+1)}\\
&\eta_i(m) = \sqrt{\frac{\Delta^2}{4} + \lambda_1^2\nu_i^2(m)}, \quad i = 1,2.
\end{aligned}
\tag{34}
$$

Using the above results we are able to discuss some statistical properties of the present system. This is seen in the forthcoming sections where we start with the atomic inversion, so we can investigate the effect of the external field on the present system.

4. Atomic inversion

We are now in a position to analyze the atomic dynamics, especially the time evolution of an important quantity, namely the atomic population inversion. The behavior of the time evolution of this quantity is used to discuss the collapses and revivals phenomenon, as a consequence of quantum interference in phase space, which is originated in the discreteness nature of the photon number distribution of the initial field [42]. It should be mentioned here that, the realization of this phenomenon has been experimentally reported in the literature [43]. The atomic inversion is introduced as the difference between the excited state and the ground state probabilities (28). Therefore, in the present section we discuss the behavior of the atomic inversion for the model which contains a two-level atom interacting with N-level atom system under the effect of an external classical field.

In Fig. 1 we plot the evolution of the atomic population subject to the scaled time $\lambda_1 t$ for different values of the coupling parameters λ_2 and λ_3 and for fixed values of the parameters $\eta = \mu = 1$, while we consider the initial angles $\vartheta = \Theta = \pi/3, \varphi = \Phi = 0$ for both the two level atom and the N-level atom. In Fig. 1(a), we display the case in which the external classical field is zero such that $\lambda_2 = \lambda_3 = 0$ (exact resonance case), in this case we can see that the atomic inversion oscillates between certain minimum and maximum values around *zero*. A typical collapse and revival, as the pure quantum mechanical phenomenon, can be observed. In the meantime and after a certain period of collapse the function $\langle\hat{\sigma}_z(t)\rangle$ shows long period of revival which is followed by several short periods of revivals. This behavior is seen for the considered periods of time, however there are no periods of collapse that appeared during the same periods. When the parameter λ_2 is taken into account and adjust $\lambda_2/\lambda_1 = 0.1$ and $\lambda_3/\lambda_1 = 0$, (off resonance case, $\Delta \neq 0$) there exists an enhancement of the energy due to the existence of the second term in (28) which in not equal to *zero*. From this

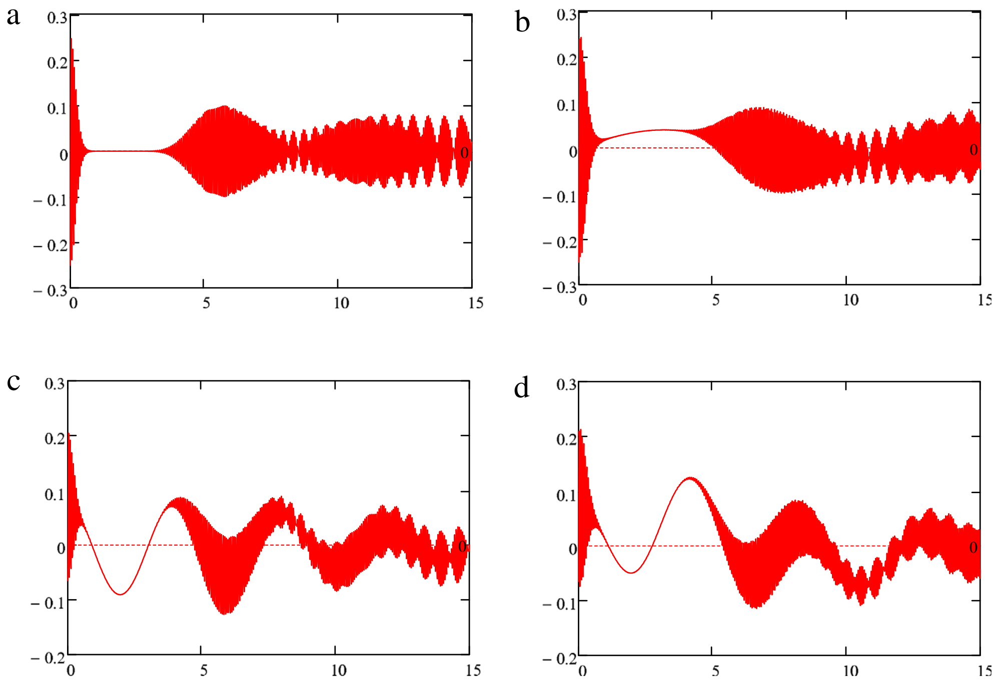

Fig. 1. The atomic population inversion against $\lambda_1 t$ for fixed parameters $\vartheta = \pi/3$, $\varphi = 0$ and $\Theta = \pi/3$, $\Phi = 0$, $\eta = \mu = 1$. (a) $\lambda_2 = \lambda_3 = 0$, (b) $\lambda_2/\lambda_1 = 0.1$ and $\lambda_3/\lambda_1 = 0$, (c) $\lambda_2/\lambda_1 = 0$ and $\lambda_3/\lambda_1 = 0.1$, (d) $\lambda_3/\lambda_1 = 0.1 = \lambda_2/\lambda_1$.

point the collapses and the revivals regions are the same as the previous case, but the function $\langle\hat{\sigma}_z(t)\rangle$ gradually oscillates around the horizontal axes. Also the behavior is observed in which the amplitude of the oscillatory behavior after the onset of interaction is decreased and shifted up, while the collapse region shows increase and attains positive values. The shifts from zero of the mean values of the period of collapses and revivals with slow vibrations are observed during the whole period of the time considered, see Fig. 1(b). When the parameter λ_3 is taken into account by setting $\lambda_3/\lambda_1 = 0.1$ while $\lambda_2/\lambda_1 = 0$, which means that $\Delta \neq 0$ and consequently the second term in (28) is not equal to *zero*, in this case the behavior of the function $\langle\sigma_z(t)\rangle$ is markedly changed. However, the symmetric behavior in the first case is completely broken after the ratio λ_3/λ_1 is taken into account. Then by comparison between Figs. 1(b) and 1(c) the effect of the parameter λ_3 is more pronounced than the parameter λ_2. Finally we study the effect of both parameters, for instance when $\lambda_2/\lambda_1 = 0.1 = \lambda_3/\lambda_1$, in this case the function decreases its amplitude while the fluctuations with the interference between the patterns still persist. In the meantime the effect of the external classical field is pronounced through the oscillatory behavior of the envelop, see Fig. 1(d).

5. Linear entropy

A simple and direct measure of the degree of entanglement between the subsystem is the linear entropy which is useful quantity to understand the degree of decoherence (coherence loss) [44]. This quantity, which can be regarded as the lowest-order approximation to the von Neumann entropy, is a good criterion to understand the purity loss of the quantum system. This parameter, which is measured by the linear entropy, has been introduced as [45]

$$P(t) = Tr[\hat{\rho}_a(t)(1 - \hat{\rho}_a(t))], \tag{35}$$

where $\hat{\rho}_a(t)$ is the atomic density operator. The above relation indicates that, the linear entropy is zero for a pure state, i.e., $\hat{\rho}_a(t) = \hat{\rho}_a^2(t)$. Consequently, the nonzero values of this indicator show the

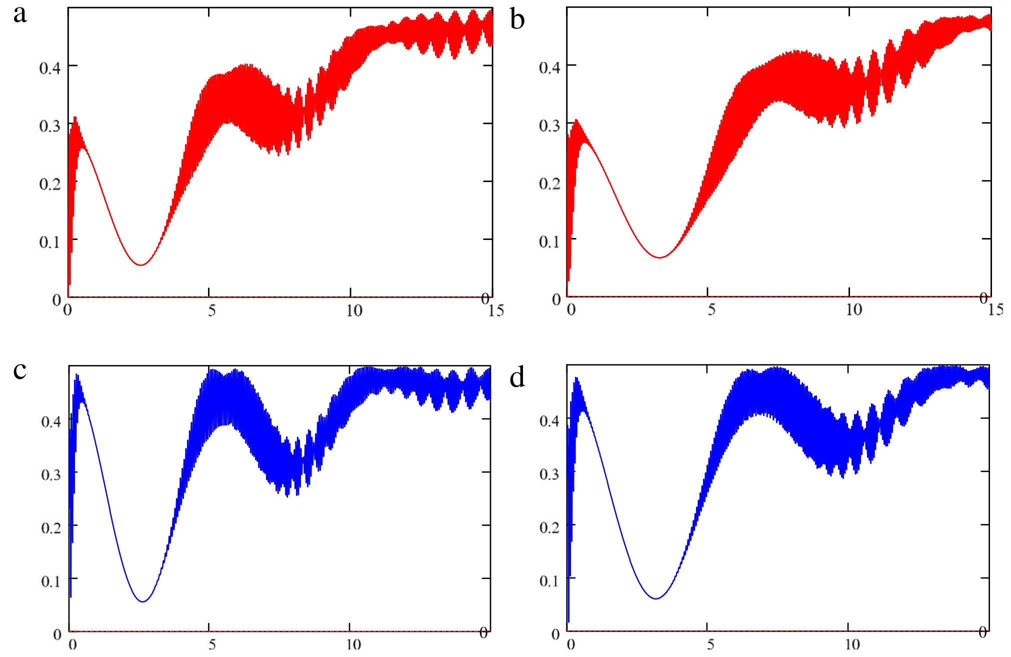

Fig. 2. The time evolution of the linear entropy changes as a function of the time $\lambda_1 t$, for fixed values of $\varphi = 0$, $\Theta = \frac{\pi}{3}$, $\Phi = 0$, $\eta = \mu = 1$. (a) $\lambda_2 = \lambda_3 = 0$, and $\vartheta = \frac{\pi}{3}$, (b) as (a) but for $\lambda_2/\lambda_1 = 0.1$ and $\lambda_3/\lambda_1 = 0$, (c) $\lambda_2 = 0$, $\lambda_3/\lambda_1 = 0.1$, while $\vartheta = \frac{\pi}{6}$ (d) $\lambda_2/\lambda_1 = 0.1$, $\lambda_3/\lambda_1 = 0.1$ and $\vartheta = \frac{\pi}{6}$.

impurity of the state of the system. Also, it may be noted that, maximal entanglement as well as the most mixed state appears whenever the idempotent defect reaches a value of 1/2.

From Eq. (28) it is easy to show that the linear entropy for the atomic state takes the following form

$$P(t) = \frac{1}{2}\left(1 - (\sigma_x^2(t) + \sigma_y^2(t) + \sigma_z^2(t)\right). \tag{36}$$

To analyze and discuss the linear entropy we use Eq. (36). However, due to the complicated expression we have plotted Fig. 2 of the function $P(t)$ against the scaled time $\lambda_1 t$, where we used the same values of the involved parameters as in the atomic inversion. In Fig. 2 we display the evolution of the linear entropy for different values of the parameters λ_2 and λ_3. In the absence of both parameters λ_2 and λ_3, we observe that the linear entropy in general satisfies the inequality $0 < P(t) < 0.5$ and consequently the function almost approaches the pure state in the middle of the collapse period showing weak entanglement. This means that the subsystems are almost nearly disentangled (at the minimum values of $P(t)$ or nearby the middle of the collapse regions in the atomic inversion, see Fig. 1(a)). This is an emphasis on the fact that, at the beginning as well as the end of the collapse region, the function $P(t)$ reaches a maximum value in which the system represented statistically correlated or entangled state, while the system approaches almost pure state at mid of the collapse time, see Fig. 2(a). On the other hand when we take the effect of the parameter λ_2 into consideration, for example $\lambda_2/\lambda_1 = 0.1$ while $\lambda_3 = 0$, the linear entropy is strongly affected, in this case the minimum value of the function $P(t)$ increases while its maximum decreases, this in addition to more regular fluctuations with some interference between patterns compared with the previous case, see Fig. 2(b). This means that the system becomes in a mixture state and displays partial entanglement. It is also noted that the degree of the mixture of the state is gradually reduced at the end of the considered period. On the other hand a different behavior is noted when we change the atomic angle ϑ, for instance we consider $\vartheta = \pi/6$, in this case the function $P(t)$ reaches maximum entanglement at different points of the considered period of time, see Fig. 2(c). The effect of λ_2 is seen in Fig. 2(d) where the function $P(t)$ increases the time to reach the maximum entanglement, in addition to more fluctuations with interference between the patterns is also observed. Note that in this case the effect of the coupling parameter λ_3 on the entanglement seems ineffective.

6. The variance squeezing

For any two observables $\hat{X}$ and $\hat{P}$ satisfying $\left[\hat{X}, \hat{P}\right] = i\hat{I}$, the Heisenberg uncertainty principle is given by $\Delta\hat{X}.\Delta\hat{P} \geq \frac{1}{2}$, where $\Delta\hat{X}$ and $\Delta\hat{P}$ show the variance of the Hermitian operators $\hat{X}$ and $\hat{P}$ which represent any two observables. In fact the variance used to define some quantum mechanical effects such as quadrature squeezing of quantum fluctuations, is not the only measure of quantum uncertainty. For instance, the entropy squeezing may be preferred instead of the variance squeezing [46]. Orłowski [47] has shown that, besides the fact that the entropic uncertainty relation is stronger than the standard uncertainty relation, the entropy (of the single observable) as well as the variance can be utilized as a measure of the squeezing of quantum fluctuations. In fact for any quantum mechanical system with two physical observables represented by the Hermitian operators $\hat{A}$ and $\hat{B}$ satisfying the commutation relation $[\hat{A}, \hat{B}] = i\hat{C}$, one can write the Heisenberg uncertainty relation in the form

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle\hat{C}\rangle|^2, \tag{37}$$

where $\langle(\Delta\hat{A})^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$. Consequently, the uncertainty relation for a two-level atom characterized by the Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$, satisfying the commutation relation $[\hat{\sigma}_x, \hat{\sigma}_y] = i\hat{\sigma}_z$ can also be written as $\Delta\hat{\sigma}_x.\Delta\hat{\sigma}_y \geq \frac{1}{2}|\langle\hat{\sigma}_z\rangle|$. Fluctuations in the component $\Delta\hat{\sigma}_\alpha$ of the atomic dipole are said to be squeezed if $\hat{S}_\alpha$ satisfies the condition

$$V(\hat{\sigma}_\alpha) = \left(\Delta\hat{\sigma}_\alpha - \left|\frac{\langle\hat{\sigma}_z\rangle}{2}\right|\right) < 0, \quad \text{where } \Delta\hat{\sigma}_\alpha = \sqrt{\langle\hat{\sigma}_\alpha^2\rangle - \langle\hat{\sigma}_\alpha\rangle^2}, \quad \alpha = x \text{ or } y. \tag{38}$$

In what follows we examine the temporal evolutions of the variances squeezing related to the present system. We have plotted Fig. 3 for the variance squeezing using the same value of the parameters as in Section 4. In Fig. 3(a) we examine the system in the absence of the external classical fields where the parameters λ_2 and λ_3 are *zero*. In this case we can observe that the squeezing occurs in the first quadratures $V(\hat{\sigma}_x)$ for a short period of time, but it is absent from the second quadratures $V(\hat{\sigma}_y)$. However, when we take the parameters $\lambda_2/\lambda_1 = 0.9$ and $\lambda_3 = 0$, we observe that the squeezing occurs in both quadratures $V(\hat{\sigma}_x)$ and $V(\hat{\sigma}_y)$ with an exchange between them. Furthermore, there is an increase in the number of fluctuations with a decrease in its minimum and consequently an increase in the amount of squeezing, see Fig. 3(b). On the other hand when we consider $\lambda_2 = 0$ and $\lambda_3/\lambda_1 = 0.9$, the squeezing amount is too small for a short period of time in both quadratures $V(\hat{\sigma}_x)$ and $V(\hat{\sigma}_y)$, but it is pronounced in the second quadrature $V(\hat{\sigma}_y)$ (compared with $V(\hat{\sigma}_x)$), see Fig. 3(c). Finally we consider the case in which $\lambda_2 = \lambda_3 = 0.9\lambda_1$, in this case the squeezing is seen in both quadratures and as the time increases the squeezing decreases. For instance the squeezing is first observed in $V(\hat{\sigma}_x)$ and consequently it starts to appear in the second quadrature $V(\hat{\sigma}_y)$, see Fig. 3(d). Here it is interesting to compare between Figs. 3(b) and 3(d) where the squeezing is pronounced in Fig. 3(b) resulting from the effect of the coupling λ_2 on the system. Thus from the above analysis it is concluded that the effect of the external classical field coupled to the N-level atom is more pronounced than the coupling to the 2-level atom.

7. The entropy squeezing

For an even N-dimensional Hilbert space, the investigation of the optimal entropic uncertainty relation for sets of $N + 1$ complementary observables with the non-degenerate eigenvalues can be

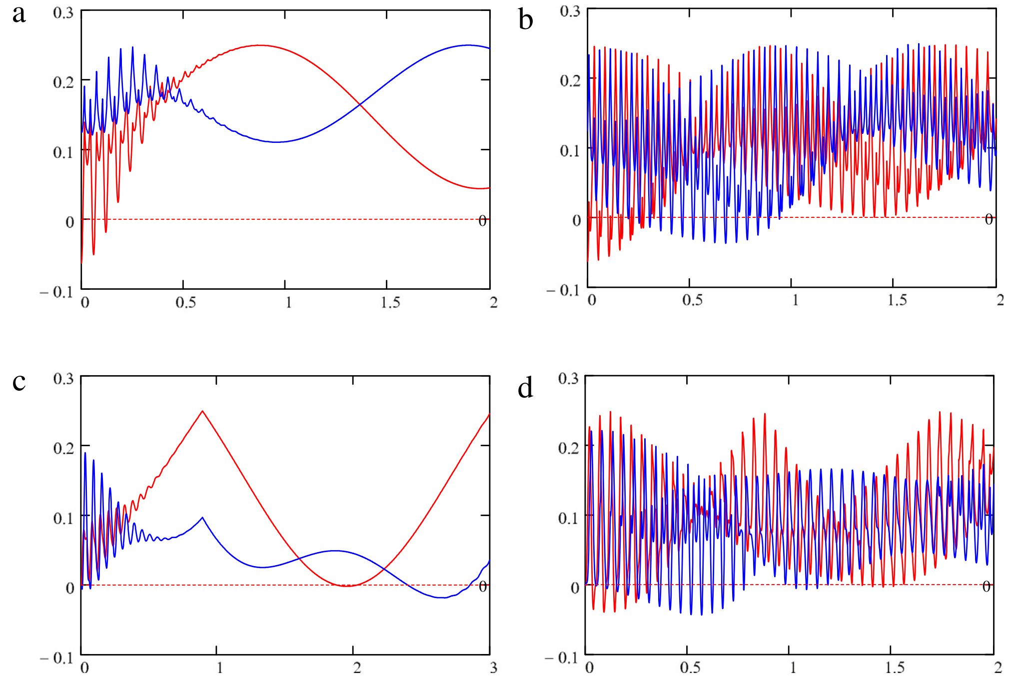

Fig. 3. The time evolution of variance squeezing changes as a function of the time $\lambda_1 t$, for fixed parameters $\vartheta = \pi/3$, $\varphi = 0$ and $\Theta = \pi/3$, $\Phi = 0$, $\eta = \mu = 1$. (a) $\lambda_2 = \lambda_3 = 0$, (b) $\lambda_2/\lambda_1 = 0.9$ and $\lambda_3/\lambda_1 = 0$, (c) $\lambda_2 = 0$ and $\lambda_3/\lambda_1 = 0.9$, (d) $\lambda_3/\lambda_1 = \lambda_3/\lambda_2 = 0.9$.

described by the inequality [48,49]

$$\sum_{k=1}^{N+1} H(\sigma_k) \geq \frac{N}{2} \ln\left(\frac{N}{2}\right) + \left(1 + \frac{N}{2}\right) \ln\left(1 + \frac{N}{2}\right), \tag{39}$$

where $H(\sigma_k)$ represents the information entropy of the variable σ_k. However, it is noted that, for an arbitrary quantum state the probability distribution for N possible outcomes of measurements of the operator σ_α is $P_j(\sigma_\alpha) = \frac{1}{2}(1 + \bar{\lambda}(\sigma_\alpha))$, $\alpha = x, y, z$, and $j = 1, 2, 3, \ldots, N$, where $\bar{\lambda}$ is a parameter which takes for a two-level atom case the value ± 1. The corresponding Shannon information entropies are then defined as

$$H(\sigma_\alpha) = -\sum_{i=1}^{N} P_i(\sigma_\alpha) \ln P_i(\sigma_\alpha), \quad \alpha = x, y, z. \tag{40}$$

In order to obtain the information entropies of the atomic operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ for a two-level spinor, we have to use the expectation value of these operators. Therefore, we can use the following expression,

$$H(\sigma_\alpha) = -\frac{1}{2}\left[\ln\left(\frac{1-\sigma_\alpha^2}{4}\right) + \sigma_\alpha \ln\left(\frac{1+\sigma_\alpha}{1-\sigma_\alpha}\right)\right], \tag{41}$$

where $\alpha = x, y, z$. For a two-level atom, we have $0 \leq H(\sigma_\alpha) \leq \ln 2$ and hence the information entropies satisfy the inequality

$$H(\sigma_x) + H(\sigma_y) + H(\sigma_z) \geq 2 \ln 2. \tag{42}$$

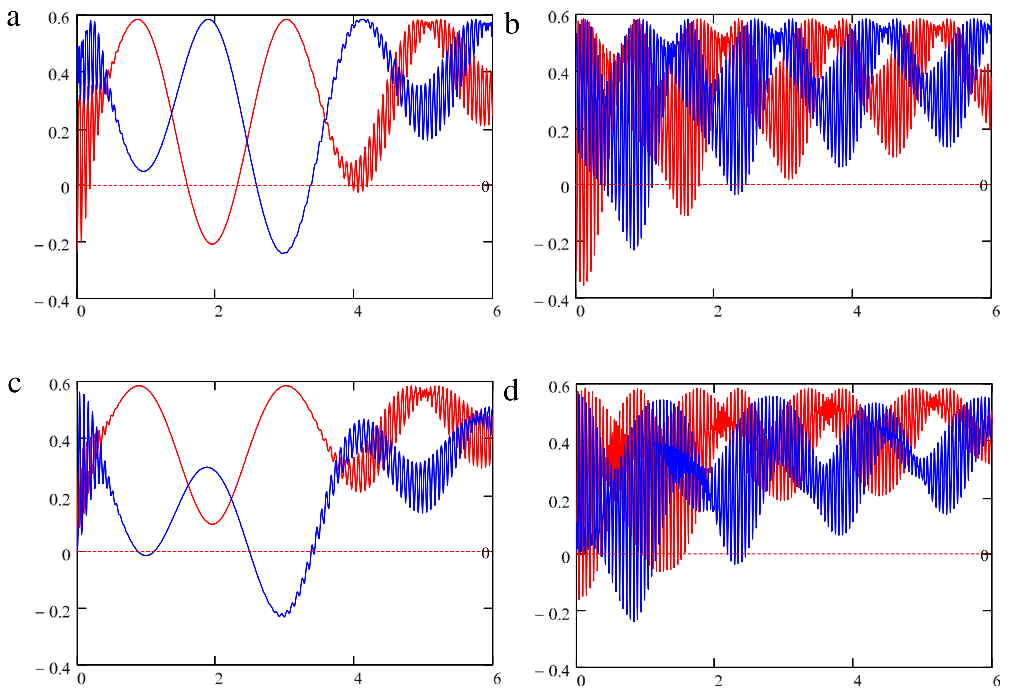

Fig. 4. The time evolution of entropy squeezing changes as a function of the time $\lambda_1 t$, for fixed parameters $\vartheta = \frac{\pi}{3}$, $\varphi = 0$ and $\Theta = \frac{\pi}{3}$, $\Phi = 0$, $\eta = \mu = 1$. (a) $\lambda_2 = \lambda_3 = 0$, (b) $\frac{\lambda_2}{\lambda_1} = 0.9$ and $\frac{\lambda_3}{\lambda_1} = 0$, (c) $\frac{\lambda_2}{\lambda_1} = 0$ and $\frac{\lambda_3}{\lambda_1} = 0.9$, (d) $\frac{\lambda_2}{\lambda_1} = 0.9$ and $\frac{\lambda_3}{\lambda_1} = 0.9$.

In other words if we define $\delta H(\sigma_\alpha) = \exp[H(\sigma_\alpha)]$, then we can write

$$\delta H(\sigma_x)\delta H(\sigma_y)\delta H(\sigma_z) \geq 4. \tag{43}$$

It is interesting to mention that the above inequality, has been established to be optimal, for more details one may consult Refs. [50–52]. Note that it is clear from the entropic uncertainty (39) the fluctuations in component $\hat{\sigma}_\alpha$ ($\alpha = x$ or y) of the atomic dipole are said to be squeezed in entropy if the information entropy $H(\sigma_\alpha)$ of σ_α satisfies the condition,

$$E(\sigma_\alpha) = \delta H(\sigma_\alpha) - \frac{2}{\sqrt{\delta H(\sigma_z)}} < 0, \quad \text{where } \alpha = x, y. \tag{44}$$

This definition for the entropy squeezing would help us to discuss this phenomenon for the present system. To do so we have plotted several figures of the entropy squeezing $E(\sigma_x)$, $E(\sigma_y)$ against the scaled time $\lambda_1 t$ for the same value of the parameters as Eq. (4).

In Fig. 4 we have fixed the value of $\eta = \mu = 1$, and considered in Fig. 4(a) the case in which $\lambda_2 = \lambda_3 = 0$. In this case, we observe the squeezing phenomenon occurring regularly in both the quadratures $E(\sigma_x)$ and $E(\sigma_y)$ as observed, however it starts first in the quadrature $E(\sigma_x)$. It is also noted that there are rapid fluctuations in both quadratures which get pronounced in $E(\sigma_y)$ after long period of time (in absence of the squeezing). When we take the parameter λ_2 into account $\lambda_2/\lambda_1 = 0.9$ and $\lambda_3 = 0$, we see that the squeezing occurs several times in the two quadratures $E(\sigma_x)$ and $E(\sigma_y)$ where its maximum amount (minimum value of the function) occurs approximately periodic in the considered time. On the other hand we see the squeezing is strongly occurring several times with rapid oscillations. Also we may refer to the regular fluctuations in all quadratures and the patterns interference which is more pronounced in the entropy squeezing factors $E(\sigma_{x,y})$ than for the variance squeezing $V(\sigma_{x,y})$ see Fig. 3(b) compared with Fig. 4(b).

As soon as we take the parameter λ_3 into consideration and adjust $\lambda_2 = 0$ and $\lambda_3/\lambda_1 = 0.9$, one can see that the regions of squeezing decreased compared with the previous case, while there is squeezing in the quadrature $E(\sigma_y)$ at two regions. The entropy squeezing factors in this case also acquire squeezing, however with amounts less than the first case see Figs. 4(a) and 4(c). In the final case we consider $\lambda_2/\lambda_1 = \lambda_3/\lambda_1 = 0.9$, where we note that for squeezing alternate periodically in Fig. 4(d). In the meantime it is noted that as the parameters λ_2, λ_3 increase more fluctuations built up, with a slight shifting down in the quadratures $E(\sigma_x)$ and $E(\sigma_y)$. This can be realized by making a comparison between the present case and previous case, see Fig. 4(c).

8. Conclusion

In the present communication we have introduced the problem of the interaction between three N-level atom and two-level atom. Under a certain condition of the Heisenberg chain the Hamiltonian model can be obtained. Two canonical transformations are used to remove two coupling parameters and hence we have managed to deal with the Hamiltonian model. The time-dependent evolution operator is used to derive the wave function which enabled us to calculate the expectation value for some dynamical operators. For instance we discussed the phenomenon of collapses and revivals through the atomic inversion where we realized that the coupling λ_2 shifts up the collapse period while λ_3 shifts it up and down. Furthermore we noted that in presence of λ_2 and absence of λ_3 the amplitude of the function after onset of interaction is more wider than the opposite case. The degree of entanglement is examined and it is found that the system is too sensitive to the variation of the atomic angle. Finally we considered the phenomenon of squeezing where two different kinds of squeezing are introduced (variance and entropy squeezing). It has been shown that both couplings are effective on this phenomenon and λ_2 displays amount of squeezing more than the amount of λ_3, where λ_3 gives longer periods of squeezing while λ_2 increases the oscillations. Investigation in this article would lead itself to discuss these phenomena in Heisenberg chains, quantum dots and star-like systems.

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